A quasi-set theory without atoms and its application to a quantum ontology of properties

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Abstract

One of the main ontological challenges posed by quantum mechanics is the problem of the indistinguishability of so-called "identical" particles, that is, particles that share the same state-independent properties. In the framework of this philosophical problem, a quasi-set theory was formulated to provide a proper metalanguage to deal with quantum indistinguishability; this theory included certain *Urelemente* called m-atoms, representing essentially indistinguishable objects. In turn, over the last two decades, the Modal Hamiltonian Interpretation proposed an ontology of properties, totally devoid of objects, where quantum systems are bundles of instances of universal properties. Therefore, the original quasi-set theory, with its m-atoms, does not adequately reflect the structure of an ontology devoid of objects. The purpose to the present article is to introduce a new quasi-set theory that does not include atoms at all: elementary items correspond to properties and are also represented by quasi-sets, which can be only numerically different. The final aim is to apply this new quasi-set theory to the MHI ontology.

Keywords: Indistinguishablity, quasi-set theory, Modal-Hamiltonian Interpretation, quantum ontology.

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> The most important thing in science is not so much to obtain new facts as to discover new ways of thinking about them.
>
> William Henry Bragg

> Each science is, first of all, a study of a phenomenology.
>
> René Thom

> All physics is impregnated with metaphysics.
>
> Mario Castagnino

1.- Introduction

One of the main ontological challenges posed by quantum mechanics is the problem of the indistinguishability of so-called "identical" particles, that is, particles that share the same state-independent properties.[1] The usual story runs as follows. Let us suppose that two identical particles are distributed over two states and we are interested in the number of possible distributions. According to the classical view, there are four possible distributions, as described by Maxwell-Boltzmann statistics. In quantum mechanics, by contrast, the permutation of identical particles does not count as a different distribution because the particles are indistinguishable. The problem consists in explaining the reason for this quantum statistical feature (Bose-Einstein and Fermi-Dirac statistics) in precise logical-ontological terms.

As repeatedly emphasised, this statistical behaviour of quantum systems challenges the category of individual of traditional metaphysics. In fact, classical individuals are endowed

[1] It is very important to emphasise from the very beginning that, when physicists talk about identical particles, the sense of "identity" is completely different from the precise meaning of the term in the logical context, where identity is the relation between two terms that refer to the same item. For this reason, here we use quotation marks to speak of "identical" particles.

with *synchronic identification*, which distinguishes an individual from all the others at a certain time, *diachronic identification*, which re-identifies an individual over time, and *transcontextual identification*, which re-identifies an individual in different contexts or situations. Quantum particles lack all three forms of identification. They cannot be synchronically distinguished from others by their properties: quantum particles cannot be labelled with conventional names. They cannot be followed as the same individual through space along time: quantum particles do not have trajectories. They cannot be re-identified when occurring in different situations: it cannot be said that a particle is the same when it appears in different collections or in different measuring arrangements. As a consequence, quantum "particles" cannot be conceived as classical individuals as characterised in traditional philosophy.

These ontological features of quantum mechanics defy the vast majority of systems of logic, including set theories, extensions of traditional logic, and deviant logics (see, e.g., Haack 1974, 1978): all of them contain symbols to represent individuals. This means that, despite the formal character of logic, the structure of most logical systems presupposes an ontology populated by individuals. As stressed by Wittgenstein (1921) in his *Tractatus*, the structure of language mirrors the structure of reality.

In the framework of this philosophical problem, the quasi-set theory $\mathfrak{Q}$ was proposed to provide a proper metalanguage to deal with quantum indistinguishability (the first printed paper is Krause 1992). In this theory, a quasi-set can include essentially indistinguishable objects. The intended ontology of the theory is an ontology as that posited by the so-called Received View about indistinguishability, according to which elementary particles cannot be regarded as individual objects (see French and Krause 2006, Bigaj 2022, French and Bigaj 2024).

In turn, over the last two decades, the Modal Hamiltonian Interpretation (MHI) of quantum mechanics has been developed (Lombardi and Castagnino 2008, Lombardi 2025): a realist interpretation belonging to the family of modal interpretations, which endows the Hamiltonian of the quantum system with a leading role in the selection of the definite-valued observables. Here we will not dwell on this aspect, but on the metaphysical side of this interpretation: the MHI proposes an ontology of properties, totally devoid of objects, where quantum systems are bundles of instances of universal properties (da Costa et al. 2013). From this perspective, the problem of indistinguishability acquires a completely different formulation.

Although the intended interpretation of quasi-set theory $\mathfrak{Q}$ is an ontology of non-individuals, thanks to its formal character it was applied to the case of the MHI ontology of properties (see Holik et al. 2022). However, the fact that the theory $\mathfrak{Q}$ includes certain *Urelemente* called m-atoms, representing non-individual objects, does not adequately reflect the structure of an ontology devoid of objects. In light of this idea, a new quasi-set theory $\mathfrak{Q}^-$ was proposed (Krause and Jorge 2024): it does not include atoms at all; elementary items[2] correspond to properties and are also represented by quasi-sets, which can be indistinguishable, that is, only numerically different.

Based on this introduction, the article is structured as follows. In Section 2, the main tenets of the quasi-set theory $\mathfrak{Q}$ will be recalled, stressing the problem of indistinguishability. Section 3 will be devoted to introduce the quasi-set theory $\mathfrak{Q}^-$, with the postulates for sets and quasi-sets. In Section 4, the theory $\mathfrak{Q}^-$ will be applied to the MHI in order to solve the problem of indistinguishability from a new perspective. Finally, Section 5 will offer some conclusions.

2.- Quasi-sets: a metalanguage for quantum mechanics

2.1.- The problem of indistinguishability

As advanced in the Introduction, one of the quantum features that have led to a deep challenge to the category of individual is the indistinguishability of the so-called "identical" particles. Indistinguishability is introduced in the formalism of standard quantum mechanics as a restriction on the set of states: non-symmetric and non-antisymmetric states are rendered inaccessible. However, this mathematical strategy does not offer a conceptual answer to the challenge, and different philosophical approaches have been proposed. For example, French (1989) has claimed that non-symmetric and non-antisymmetric states are ontologically possible and only physically inaccessible: indistinguishability is a physical fact, not an ontological condition. However, this move has an unavoidable ad hoc flavour, since positing inaccessible states amounts to introducing a surplus structure in the formalism (see Redhead, and Teller 1992). Van Fraassen (1985), in turn, has suggested that individuality can be recovered while retaining quantum statistics by giving up the equiprobability of the different distributions of quantum particles in quantum states. Another approach to the problem has been based on the idea of "weak discernibility" (Saunders 2003, Muller and Saunders 2008):

[2] Here the terms 'item' or 'entity' refer to whatever that may exist: object, property, event, process, etc.

in the case of two fermions in a singlet state, the relation 'having the opposite direction of each spin component with respect to …' that each fermion has with respect to the other is sufficient to establish numerical distinction between the objects, even if they are indistinguishable with respect to their monadic and relational properties (for an analogous argument for bosons, see Muller and Seevinck (2009). French and Krause (2006) have rejected this strategy by arguing that it involves circularity (see also Krause 2010): in order to appeal to that kind of relations, the discrimination between the related objects must be presupposed in advance and, as a consequence, their numerical difference was already assumed.

The problem of indistinguishability in quantum mechanics has generated extensive philosophical debates. Here we will not review this complete field of discussion, but will only focus on the approaches that emphasise the limitation of standard set theory, which includes individual elements, to deal with an ontology that defies the ontological category of individual. In particular, we will consider quasi-set theory in two versions, in both cases conceived as a metalanguage that provides the formal tools to speak meaningfully when the elemental items are not individuals.

2.2.- The theory $\mathfrak{Q}$

In standard set theory, the elements of sets are always distinct from each other. The word 'distinct', as employed in this context, requires explanation. Distinct is the opposite (or the negation) of 'identical'. 'Identity', in the philosophical terminology, is applied to terms that refer to the very same item: identity is sameness. So, 'distinct' means different; but different in what sense? In traditional metaphysics, two items are always different in the sense that they are not the same. Moreover, according to the Principle of Identity of Indiscernibles (PII), two items always differ regarding some properties: they cannot be absolutely indistinguishable. This is the assumption defied by quantum mechanics: quantum particles may be indiscernible or indistinguishable and yet be a plurality. It is precisely to account for this quantum feature that the quasi-set theory $\mathfrak{Q}$ was originally formulated. It is important to recall that the theory does not consider other possible distinguishing characteristics such as some form of *substratum* or *haecceity* (see Teller 1998).

The motivations for quasi-set theory come from two basic sources: (i) according to Schrödinger, identity (in the sense of sameness) makes no sense for quantum items: they are non-individuals (see Post 1963) since identity is a typical feature of the traditional category of individual. On this basis, (ii) the lack of identity should be attributed to quantum entities right from the start (see French and Krause 2006) and not a posteriori as can be done in standard mathematics.[3] Indeed, this a posteriori move leads to a "fake" non-individuality introduced by hand, and not to a non-individuality representing a fundamental metaphysical fact.

In this theoretical context, quasi-set theory was proposed to deal with collections of completely indiscernible objects. A quasi-set (*qset* hereafter) may have elements that cannot be discerned in any way and to which the Standard Theory of Identity (STI) does not apply. Some qsets can be constituted by discernible entities, so they behave as standard sets of the Zermelo-Fraenkel (ZF) set theory. In other words, standard sets can be viewed as particular cases of qsets.[4]

The first theory of quasi-sets (Krause 1992), the theory $\mathfrak{Q}$, was a theory with atoms (*Urelemente*), that is, entities that can be elements of qsets but which (in principle) do not have elements and are not the empty set. This is in accordance with set theories with atoms, such as the ZFA set theory (Zermelo-Fraenkel with atoms; see Suppes 1972, Jech 2003). However, by contrast to ZFA, theory $\mathfrak{Q}$ admits the existence of two kinds of atoms:

– *M-atoms*, which work as the *Urelemente* in the ZFA set theory. They are entities endowed with identity conditions, which correspond to classical objects.
– *m-atoms*, which play the role of quantum entities. They may be indiscernible without being the same item.

Collections of M-atoms and/or m-atoms are qsets (if they satisfy the rules of formation of $\mathfrak{Q}$). Sets are those qsets whose transitive closure does not contain m-atoms; in other words, they are qsets constructed in the "classical part" of the theory. When the theory is restricted to this classical part, it becomes equivalent to ZFA, and to ZFC (Zermelo-Fraenkel without atoms) if M-atoms are also removed.

The identity conditions of M-atoms are ruled by the STI, formulated as in ZFA: identity is represented by a binary predicate '=' that satisfies reflexivity, substitutivity, and extensionality (see French and Krause 2006). By contrast, m-atoms can be indiscernible even

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[3] For example, if the discourse is restricted to a deformable (non-rigid) structure, it can simulate indiscernibility by invariance under (non-trivial) automorphisms (see Krause and Coelho 2005),

[4] Recall that Cantor "defined" the concept of set by stating that "[b]y an 'aggregate' (*Menge*) we are to understand any collection into a whole (*Zusammenfassung zu einem Ganzen*) *M* of definite and separate objects *m* of our intuition or our thought. These objects are called the 'elements' of *M*." (Cantor 1955; see discussion in French and Krause 2006: Chap.6). The axiomatisation of set theory, started by Zermelo, formalises this idea.

if they are not identical. If x and y are *indiscernible* m-atoms, the relation is represented as '$x \equiv y$'. A qset may have a *quasi-cardinal*, something that is intended to express the quantity of elements it has, so that it resembles the cardinals of sets. If x is a qset, then $qc(x)$ expresses its quasi-cardinal, or simply *q-cardinal* for short.

The main motivation for the $\mathfrak{Q}$ theory is the interpretation that deprives quantum entities (electrons, protons, photons, atoms, etc.) of identity. In some situations, m-atoms cannot be distinguished from each other in any way, but nevertheless they can be considered numerically distinct. This fact is contrary to STI: according any mathematical theory encompassing STI and agreeing with the PII, given two entities whatever, they are distinct, and this entails that there exist (even if only in principle) a property of just one of them and not of the other. In the interpretation assumed here, by contrast, quantum entities can sometimes be completely indiscernible, such as bosons in a bosonic condensate (see, e.g., Ketterle 1999). Fermions, in turn, cannot be in the same quantum state due to Pauli's Exclusion Principle; nevertheless, the whole situation may be such that it is not possible to discriminate which element is which. For example, in a neutral helium atom, the total pure state is a superposition of two spin states: a state with one electron with spin UP and the other with spin DOWN in some direction, and a state with the contrary situation. But 'having spin UP' (or DOWN) cannot be considered as a property of the electrons: in the superposition state, they do not have particular properties; only the entire system has total spin zero. Although after a measurement of the spin in the chosen direction it can be said that one of the electrons has spin UP and the other has spin DOWN, it makes no sense to say something like 'electron Peter has spin UP' as if 'Peter' were a proper name. Due to this fact, it can be meaningfully said that even fermions are indiscernible. Summing up, as dalla Chiara and Toraldo di Francia (1993) claim, the quantum realm is a world of anonymity; proper names do not act as rigid designators.

Even if a qset has only m-atoms as elements, this does not imply that they are all completely indiscernible. In fact, m-atoms can differ by some properties, as electrons differ from protons and neutrons.[5] Two m-atoms are of the same kind when they represent quantum particles with the same state-independent properties: only m-atoms of the same kind can be indiscernible. In turn, two qsets are indiscernible when they are composed of elements of the

[5] Furthermore, quantum entities of the same kind can also be discerned in some situations, say, by different polarisations. So, it is not assumed that all entities of the same kind are indiscernible. The problem, as said already, is that their 'mock' particularisation does not act as something that can ascribe them the epithet of 'being an individual'.

same kind and they have the same q-cardinality. For example, two molecules of sulfuric acid are indiscernible, '$H_2SO_4 \equiv H_2SO_4$': no identity of the components is required; there are two absolutely indistinguishable atoms of H in each molecule. This means that $\mathfrak{Q}$ theory does not include an Axiom of Extensionality in the usual sense, but an Axiom of Weak Extensionality, which informally states that qsets of indiscernible elements (elements of the same kind) with the same q-cardinal (the same number of elements) are indiscernible.

3.- The theory $\mathfrak{Q}^-$

3.1.- Presentation of the theory

Theory $\mathfrak{Q}^-$ is a modification of theory $\mathfrak{Q}$: the minus of its name indicates that it does not include atoms. As explained in the Introduction, the purpose of this formulation is to have a formal resource to describe an ontology of properties deprived from objects. In this subsection, theory $\mathfrak{Q}^-$ will be introduced.

Let us consider an intended universe $\mathfrak{B}$ (*Bereich*, as called by Zermelo 1967), whose elements are of two kinds: qsets, some of which are sets, and finite q-cardinals. The language $\mathcal{L}$ of the theory comprises the following categories of primitive symbols:

(i) Standard logical symbols for negation, '$\neg$', implication '$\rightarrow$', and universal quantifier '$\forall$' are primitive. The other standard propositional connectives, '$\wedge$', '$\vee$', and '$\leftrightarrow$', as well as the existential quantifier '$\exists$', are defined as usual. Improper symbols such as parentheses and commas are also introduced.

(ii) "Individual" variables of two kinds: $x, y, z, \ldots$ are metavariables for qsets, and $m, n, p, \ldots$ are metavariables for finite q-cardinals. Qsets will be denoted with '$[\,]$', analogously to '$\{\}$' for sets in ZF. We remark that 'finite q-cardinal' is a primitive notion.

(iii) A unary predicate symbol 'S' to designate sets.

(iv) The equality symbol, '$=$'.

(v) The indiscernibility symbol, '$\equiv$'.

(vi) A unary functional symbol 'qc' for arbitrary or "general" q-cardinals. Finite q-cardinals are particular cases of general q-cardinals.

(vii) The symbol for the membership relation, '$\in$'.

(viii) A binary predicate symbol 'K' that connects a finite qset with its finite q-cardinality.

(ix) Three specific primitive symbols to denote finite q-cardinals: an individual constant '$\overline{0}$', a unary functional symbol 's' and two binary functional symbols, '$\oplus$' and '$\otimes$'.

Before proceeding further, it is worth noting that, despite that variables $x, y, z, \ldots$ are called 'individual variables', they may range over a domain whose elements are non-individuals, that is, items without identity conditions, which constitute the "phenomenology" of the theory. In particular, in the intended interpretation of theory $\mathfrak{Q}^{-}$, those items are not objects but properties. In fact, "[a]n ontological domain populated exclusively by properties and non-individual bundles of properties cannot be adequately apprehended by any language that includes individual constants and variable." (Holik et al. 2022; 12). However, starting with a language that does not include individual variables is not an easy task. Tarski tried to do it when he proposed his Calculus of Relations (Tarski 1941) and his Set Theory Without [individual] Variables (Tarski and Givant 1987). In these works, he started with a standard language with individual variables and variables for relations. Then, he selected some of the theorems about relations and started again, but now dispensing with the individual variables and taking the chosen theorems as axioms; in this way, he claims to have arrived at a calculus of relations. Of course, this is a trick. In theory $\mathfrak{Q}^{-}$ the strategy is different: the individual variables range over qsets, which can be conceived of as extensions (or "semi-extensions") of predicates.

The terms of $\mathcal{L}^{-}$ are the individual variables, the individual constant 0, the expressions of the form $qc(x)$, $s(m)$ (which will be abbreviated by simply 'sm'), $m \oplus n$, and $m \otimes n$.

The formulas are defined recursively as usual, with the proviso that the use of the predicate K is such that only expressions of the form $K(x, m)$ are formulas. If $S(x)$, then we say that x is a set. Restricted quantifiers are used; therefore, given a formula φ, $\forall_S x\varphi$ stands for $\forall x\left(S(x) \to \varphi\right)$, whereas $\exists_S x\varphi$ abbreviates $\exists x\left(S(x) \wedge \varphi\right)$.

The logical axioms, that is, the postulates (axiom schema and inference rules) of the underlying logic are the following:

(1) $\alpha \to (\beta \to \alpha)$

(2) $(\neg\alpha \to \neg\beta) \to ((\neg\alpha \to \beta) \to \alpha)$

(3) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$

(4) $\forall x(\alpha \to \beta) \to (\alpha \to \forall x\beta)$, where x does not appear free in α.

(5) $\forall x \alpha \rightarrow \alpha(t)$, where t is a term free for x in α.[6]

Remark: If α is $K(x,m)$, then the terms must agree with the kind of entity they represent, that is, qsets for x and q-cardinals for m.

(6) The inference rules are Modus Ponens and Generalisation, that is,

$$\frac{\alpha, \alpha \rightarrow \beta}{\beta} \qquad \frac{\alpha}{\forall v \alpha}$$

where v is a variable for qsets or for q-cardinals, and the standard conditions are observed.

(7) $\forall_S x\,(x = x)$

(8) $\forall_S x \forall_S y\big(x = y \rightarrow (\alpha(x) \rightarrow \alpha(y))\big)$

(9) $\forall m\,(m = m)$

(10) $\forall x \forall y\big(m = n \rightarrow (\alpha(m) \rightarrow \alpha(n))\big)$

(11) Standard conditions are imposed for (8) and (10), given the above restrictions mentioned in the case of $K(x,m)$.

The postulates say that, for sets and q-cardinals, the standard postulates of the First-order calculus with identity hold, but for qsets that are not sets, the same calculus without identity holds. Since the notion of identity does not apply to qsets, the Principle of Identity under the form $\forall x\,(x = x)$ neither applies to them. Since this principle is also called Reflexive Rule of Identity, this logic belongs to the realm of non-reflexive logics (see French and Krause 2006, da Costa and Bueno 2009).

Finite q-cardinals obey the postulates of first-order Peano Arithmetics. The reason for taking them "separately", as a step theory, is to avoid the compromise of finite q-cardinals with ordinals. Let us explain this point more precisely. Usually, cardinals are defined through the notion of ordinal: cardinals are particular cases of ordinals (this is von Neumann's approach; there are other ways of introducing cardinals, but they will be not discussed here: see Krause and Jorge 2024). Then, when a cardinal is attributed to a collection, an ordinal is also attributed to it; this leads to the identification of the elements of the collection by means of their order. But this is a difficulty for the hypothesis that there are collections (qsets) of indiscernible elements. For this reason, finite q-cardinals are taken to be $\overline{0}$, $s\overline{0}$, $ss\overline{0}$, etc. Of course, it is easy to notice that these "numerals" have the same models as the standard natural

[6] As usual, $\alpha(x_1, \ldots, x_n)$ means that the free variables of α are among the x_i; therefore, $\alpha(t_1, \ldots, t_n)$ is obtained by the substitution of x_i by t_i.

numbers defined (in the sense of von Neumann) as ordinals, namely, $\varnothing$, $\{\varnothing\}$, $\{\varnothing,\{\varnothing\}\}$, etc. However, they are not the same kind of mathematical object. So, finite q-cardinals are not ordinals, and when a finite q-cardinal is attributed to a qset of indiscernible items, this does not presuppose that its elements are discernible or have identity.

The postulates for finite q-cardinals are:

(1) $\forall m\,(\overline{0} \neq sm)$

(2) $\forall m \forall n\,(sm = sn \rightarrow m = n)$

(3) $\forall m\,(\overline{0} \oplus m = m)$

(4) $\forall m \forall n \big(m \oplus sn = s(m \oplus n)\big)$

(5) $\forall m\,(\overline{0} \otimes m = \overline{0})$

(6) $\forall m \forall n \big(m \otimes sn = (m \otimes n) \oplus m\big)$

(7) $\forall m\,(m^{\overline{0}} = s\overline{0})$

(8) $\forall m \forall n\,(m^{sn} = m^{n} \otimes m)$

(9) If P is a property that applies to q-cardinals, then

$$P(\overline{0}) \wedge \forall m \big(P(m) \rightarrow P(sm)\big) \rightarrow \forall m\, P(m)$$

Definition 1 (Finite q-cardinals): *The finite q-cardinals are abbreviated this way:* $\overline{1} := s\overline{0}$, $\overline{2} := s\overline{1}$, etc.

Consequently, propositions as '$K(x,\overline{1})$', '$K(x,\overline{2})$' can be used to express that a qset x has one or two elements, respectively, even without identification. In general, '$K(x, s^{n}\overline{0})$' can be written.

3.2.- Postulates for qsets

In this section, the postulates for qsets will be presented in order to keep the paper self-contained, but they will be not discussed in depth. For details, the reader can see Krause and Jorge (2024).

The basic idea of theory $\mathfrak{Q}^-$ consists in supposing a domain of qsets, some of which are classified as sets and are ruled by the postulates of ZFC. The qsets that are not sets are called *pure*, and they do not obey all the postulates of ZFC, in particular, STI.[7]

Definition 2 (Pure qsets): *A pure qset is defined by means of the primitive predicate S as $P(x) := \neg S(x)$*

Thus, some specific axioms of $\mathfrak{Q}^-$ are (restricted quantifiers will be used):

(1) The ZFC axioms for sets, that is, for items that satisfy the predicate S.

(2) $\forall_S x \forall_S y \left(x \equiv y \to x = y \right)$

(3) $\exists_P x \left(x \equiv x \right) \wedge \forall x \left(x \equiv x \right)$

(4) $\forall x \forall y \left(x \equiv y \to y \equiv x \right)$

(5) $\forall x \forall y \forall z \left(x \equiv y \wedge y \equiv z \to x \equiv z \right)$

(6) $\forall y \left(S(y) \to \forall x \left(x \in y \to S(x) \right) \right)$
This axiom says that the members of a set are also sets.

(7) [*Separation Schema*] Let α be a formula $\mathcal{L}^-$. Hence,
$\forall x \forall m \Big(K(x,m) \to \exists y \forall z \big(z \in y \leftrightarrow \exists w \forall n \left(w \in x \wedge z \equiv w \wedge K(y,n) \wedge n \leq m \right) \wedge \alpha(z) \big) \Big)$
The qset y is denoted as $\left[z : \alpha(z) \right]_x$, with the proviso that its q-cardinality may be not greater than the q-cardinality of x. Furthermore, $n \leq m$ should be read as $\exists p \, (m = n \oplus p)$.

(8) $\forall x \, K(x, \overline{0}) \leftrightarrow \neg \exists y \, (y \in x)$
This axiom says that the q-cardinality of any empty qset is zero. Notice that the theory is compatible with the existence of more than one empty qset; in fact the proof of uniqueness of the empty set cannot be reproduced in $\mathfrak{Q}^-$ as a consequence of the Weak Extensionality Axiom, which will be introduced below.

(9) [*Union*]
$$\forall x \forall y \exists z \forall w \Big(w \in z \leftrightarrow \exists w' \exists w'' \big(\left(w' \in x \wedge w \equiv w' \right) \vee \left(w'' \in y \wedge w \equiv w'' \right) \big) \wedge \forall m \forall n \left(K(x,m) \wedge K(y,n) \to K(z,p) \wedge p \leq m \oplus n \right) \Big)$$

(10) $\forall x \forall y \forall m \left(K(x,m) \wedge K(y, \overline{1}) \wedge \neg \exists z \left(z \in x \wedge z \in y \right) \to K(x \cup y, sm) \right)$

[7] The qsets called 'pure' here were called 'legitimate' in Krause and Jorge (2024). In the framework of that article, 'pure' is used to refer to qsets that do not have atoms, and 'legitimate' is used to refer to qsets that do not satisfy the ZF axiomatics. In the present theory $\mathfrak{Q}^-$, no qset has atoms, so all legitimate qsets are pure.

This postulate says that if we "add" an element to a qset which is not "already there", the q-cardinal of the qset increases by one unity

Definition 3 (Weak singleton): *Let* w *be a qset and* x*,* y *be elements of* w*. By means of the condition* $\alpha(t) \leftrightarrow t \in w \wedge t \equiv x$*, the qsets that are indiscernible from* x *and that belong to* w *can be obtained through the Separation Schema and they will be denoted by* $[x]_w$.

(11) [*Pairing*] $\forall_P w \forall x \forall y \big(x \in w \wedge y \in w \rightarrow \exists z \forall t \big(t \in z \leftrightarrow \big(t \in [x]_w \vee t \in [y]_w \big)\big)\big)$
The case in which w is a set was already considered in the Pair Axiom of ZFC. It is clear that, in principle, nothing indicates the q-cardinality of the qset z, except that it does not exceed the q-cardinality of w. If $x \equiv y$, it may even be $\overline{1}$.

(12) [*Weak Extensionality Axiom*]
$\forall x \forall y \forall n \Big(\big((\forall z \in x/_{\equiv})(\exists w \in y/_{\equiv}) \big) \big(K(z,n) \rightarrow K(w,n) \big) \wedge \forall u \forall v \big(u \in z \wedge v \in w \rightarrow u \equiv v \big) \wedge$
$\big(\forall w \in y/_{\equiv}\big)\big(\exists z \in x/_{\equiv}\big)\big(K(w,n) \rightarrow K(z,n) \wedge \forall u \forall v \big(u \in v \wedge v \in z \rightarrow u \equiv v \big)\big) \rightarrow x \equiv y \Big)$
This axiom states that, if the qsets x and y have "the same number" of indiscernible elements (given by q-cardinals), then x and y are indiscernible. This allows us to write '$C_7H_8N_4O_2 \equiv C_7H_8N_4O_2$', i.e., two theobromine molecules (the molecule associated with cocoa) are indiscernible, which seems very reasonable. This axiom is primarily responsible for the fact that, in $\mathfrak{Q}^-$, there can exist indiscernible, but not identical, empty qsets. If the identity relation $(=)$ is used in (12) instead of the indiscernibility relation, then the equivalence classes of $x/_{\equiv}$ become unitary (the same with $y/_{\equiv}$) and the Axiom of Weak Extensionality reduces to the Axiom of Extensionality of ZFC; in that case, $x \equiv y$ collapses to $x = y$.

Definition 4 (Weak Membership): $x \in^* y := \exists w \big(w \in y \wedge w \equiv x \big)$

It can be proved that, if x or y are sets, then $x \in y \leftrightarrow x \in^* y$. By means of Definition 4, weak inclusion can be defined.

Definition 5 (Weak Inclusion):
$x \subseteq^* y := \forall z \big(z \in x \rightarrow z \in^* y \big) \wedge \forall m \forall n \big(K(x,m) \wedge K(y,n) \rightarrow m \leq n \big)$
If $x \subseteq^* y$, then x is a sub*-qset of y.

Definition 6 (Cloud): *Let* w *be a qset and let* $x \subseteq y$*. The cloud of* x *relative to* w *is the qset* $N(x,w)$ *such that* $N(x,w) = [y \in w : \exists z \in x \wedge z \equiv y]$

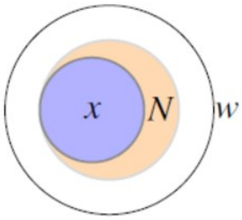

Figure 1: A qset x and its cloud $N(x,w)$ relative to the qset w.

(13) $\forall x \exists y \left(\forall z \left(z \in y \leftrightarrow z \subseteq x \right) \right)$

If x is a qset, then there exists a "power" qset, denoted by $\mathcal{P}(x)$, that has the sub-qsets of x as elements.[8]

(14) $\forall x \left(K(\mathcal{P}(x), \overline{2}^n) \leftrightarrow K(x,n) \right)$

This axiom makes it possible to consider the number of sub-qsets in a qset x in the same way as the case of sub-sets of sets: if the q-cardinal of x is n, then it is consistent with the theory to suppose that there are n unitary sub-qsets, C_2^n sub-qsets with two elements (the number of combinations of the n elements taken two by two), etc.

(15) $\forall x \forall n \left(K(x,n) \rightarrow \left(\forall p \left(p \leq n \right) \rightarrow \exists y \left(y \subseteq x \wedge K(y,p) \right) \right) \right)$

This axiom means that, if the q-cardinal of a qset x is n, then x has a sub-qset of q-cardinality p for any $p \leq n$.

The most important point for the present proposal up to here is that, given a qset x such that $K(x,n)$, it can be consistently said that there is at least one sub-qset of x with q-cardinal one. If all the elements of x are indiscernible from each other, then all such "unitary" sub-qsets will be indiscernible, according to the Axiom of Weak Extensionality. With the axioms up to this point, it can be inferred that, if a qset has a q-cardinal $n \geq \overline{1}$, then there are n sub-qsets of it with unitary q-cardinals, which will be called *strong singletons*.

Definition 7 (Strong Singleton): *Let x be a qset with non-zero q-cardinal n. If $y \in x$, it will be said that a qset is a strong singleton of y relative to x, denoted by $[\![y]\!]_x$, if its q-cardinal is $\overline{1}$.*

(16) $\forall x \forall n \left(K(x,n) \rightarrow \left(\forall y \subseteq x \right) K(y,\overline{1}) \rightarrow K(x \setminus y, n - \overline{1}) \right)$

where '$K(x \setminus y, n - \overline{1})$' means '$K(x \setminus y, m) \wedge n = sm$'. In this axiom it is accepted that an element can be "removed" from a qset x by admitting the existence of the qset y,

[8] Axiom (13) could be formulated by using $\subseteq^*$.

even though the removed element cannot be identified. Axiom (16) means that, if an element of a qset is removed, the q-cardinality of that qset decreases by one unit.

Theorem1 (Symmetry under Permutations): *Let* x*,* y, z *be qsets such that* $x \subseteq y$*, and let* $w \in x$*. If* $z \in y$ *and* $z \equiv w$*, but* $z \notin x$*, then* $\left(x \setminus [\![w]\!]_x\right) \cup [\![z]\!]_y \equiv x$ (see proof in Krause and Jorge 2024).

This theorem, which is very relevant to distinguishing theory $\mathfrak{Q}^-$ from ZFC, says that, if we "exchange" an element of a qset x for an indiscernible one, the resulting qset is indiscernible from the original. This result allows expressing what happens in an He atom: if an electron is first emitted, leading to a cation He^+, and then the cation captures an electron, leading to a neutral atom, it makes no sense to ask whether the captured electron is the same as the emitted electron, or whether the new neutral atom is the same as the original.

(17) [*Selection of Indiscernibles*]

$$\forall_Q x\left(\forall y \forall z\left(y \in x \wedge z \in x \rightarrow y \cap z = \varnothing \wedge \neg\left(y = \varnothing\right)\right) \rightarrow \right.$$
$$\left. \exists_Q u \,\forall y \forall v\left(y \in x \wedge v \in y \rightarrow \exists_Q w\left(w \subseteq [v]_y \wedge qc(w) = \overline{1} \wedge w \cap y \equiv w \cap u\right)\right)\right)$$

Postulate (17) was called 'Axiom of Choice' in French and Krause 2006 and 'Axiom of Quasi-Choice' in de Barros et al. (2024). However, this terminology has led to confusions that generated unfounded criticisms. Therefore, in order to avoid any association with the standard Axiom of Choice, here the name 'Selection of Indiscernibles' is used. According to this axiom, some selections are possible in theory $\mathfrak{Q}^-$. Let us consider a qset whose elements are non-empty and disjoint qsets, say, a collection of three collections, one of electrons, one of protons, and one of neutrons. The idea of Axiom (17) is that there exists a "selection" qset whose elements are indiscernible from the respective elements of each of the elements of the given qset, i.e., a collection with one electron, one proton, and one neutron. It is not necessary that the elements of the selection qset are elements of the qsets of the given qset, as is the case with the usual Axiom of Choice. The important point is that the selection qset has elements indiscernible from the elements of those qsets.

(18) [*Regularity*] $\forall x\left(\exists y\left(y \in x\right) \rightarrow \exists y\left(y \in x \wedge \neg \exists z\left(z \in x \wedge z \in y\right)\right)\right)$

This axiom states that every non-empty qset x has an element that has no elements in common with x.

4.- Application to a quantum ontology of properties

4.1.- The Modal-Hamiltonian Interpretation of quantum mechanics

A quantum ontology of properties was originally suggested in the first presentation of the Modal-Hamiltonian Interpretation of quantum mechanics (Lombardi and Castagnino 2008), but it acquired a more developed form some years later (da Costa et al. 2013, da Costa and Lombardi 2014). Although introduced in the framework of a specific interpretation, it should be emphasized that this ontology can be adopted in the context of a different realist interpretation. For this reason, the present article will not analyse the MHI in general but only the structure of the proposed ontology.

Properties, being necessary for classification, are included in almost any ontological picture. They have traditionally been conceived as *universals*, sometimes as transcendent or *ante res*, and sometimes as immanent or *in rebus*. Nevertheless, in both cases a universal has multiple instances and is fully present in each of its instances. As Duns Scotus asserted, the peculiarity of universals is that they are "one-in-many" (see, e.g., King 1992): a universal is one, for example roundness, but it is instantiated in a multiplicity of cases, roundness in this case, roundness in that case. The instances of a universal property are many but absolutely indistinguishable: they differ *solo numero*; the roundness of a billiard ball and the roundness of a water drop are both instances of the universal roundness, but trying to distinguish them as different properties makes no sense. The fact that instances of a universal are only numerically different offers an adequate basis to provide an ontological answer to the problem of quantum indistinguishability.

In 20th century metaphysics, a new approach to properties entered the scene: properties as *tropes*, which are particular properties, such as the particular shape, weight, and texture of an object (see Maurin 2018). Although tropes may be absolutely similar, they are neither absolutely indistinguishable nor only numerically different, precisely because they can be individuated and distinguished by their space-time position, by the object to which they apply, or because the distinction between them is taken to be primitive. For this reason, an ontology of tropes for quantum mechanics would face the same difficulties as an ontology of objects, since in both cases they are distinguishable items.

Another distinction in terms of properties is the traditional difference between *determinables* and *determinates*, that is, properties that are in a distinctive specification relation usually called '*determination*' (see Wilson 2022). For example, colour is a determinable having red, blue, and other specific shades of colour as determinates; mass is a determinable having specific mass values as determinates. It is important to emphasise that the determinable-determinate relationship should not be confused with the universal-instance relationship. The latter is the relation between a universal property and its many instances: for example, the universal property colour has countless instances of coloured items. The first, by contrast, is the relation between a property and other more specific properties that are specifications of it: for example, the determinable colour has red, green, yellow, etc. as its determinates.

Let us now consider the ontological category of individual objects: is an individual a substratum supporting properties or a mere "bundle" of properties? (see, e.g., Robinson 2021, Rettler and Bailey 2022). The conception of an individual as a substratum acting as a carrier of properties has pervaded the history of philosophy: it is present in different forms, for example, in Aristotle's "primary substance" and in Locke's "substance in general". However, following Hume's rejection of the notion of substance, many philosophers belonging to the empiricist tradition have regarded the postulation of a characterless substratum as a metaphysical abuse and have adopted some version of the bundle theory. According to this view, an individual is nothing but a bundle of properties: properties have metaphysical priority over individual and are, therefore, the fundamental items of the ontology.

On the basis of these metaphysical preliminaries, the general structure of the MHI ontology can be introduced.

I) *Type-properties*

The term 'observables' in quantum mechanics refers to quantifiable magnitudes of physical relevance, which are mathematically represented by self-adjoint operators. Ontologically, they correspond to items belonging to the category of property, here conceived as universals. In turn, since in general observables have different values, their ontological counterparts are determinable properties, which here will be referred to as "type-properties". Therefore, the ontological counterparts (i) of general physical magnitudes are *universal type-properties* (*U-type-properties*), symbolized as $\langle A \rangle$, and (ii) of observables are *instances of universal type-properties* (*I-type-properties*), symbolized as $[A]$. An example of U-type-property is spin in the *x*-direction $\langle S_x \rangle$, which can be instantiated as the spin in the *x*-direction $[S_x]$ in this particular system. If the physical, the ontological, and the mathematical languages are distinguished, it should be said that (a) each I-type-property of a U-type-property is physically represented by an observable A belonging to a certain space of observables $\mathcal{O}$ ($A \in \mathcal{O}$), and (b) each observable is mathematically represented by a self-adjoint operator $\hat{A}$ belonging to a space of self-adjoint operators $\hat{\mathcal{O}}$ ($\hat{A} \in \hat{\mathcal{O}} = \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the von Neumann algebra of linear bounded operators on the Hilbert space $\mathcal{H}$).

II) *Case-properties*

Since a physical observable is a quantifiable magnitude, it has different possible values, which are mathematically represented by the eigenvalues of the corresponding self-adjoint operator. Their ontological counterparts are determinate properties, which here will be referred to as "case-properties" of the corresponding type-property. At this level it is also necessary to distinguish between *universal case-properties* (*U-case-properties*) and *instances of universal case-properties* (*I-case-properties*): given an I-type-property $[A]$ of a U-type-property $\langle A \rangle$, (i) its I-case-properties will be symbolized as $[a_i]$, (ii) each one of which is an instance of the respective U-case-property symbolized as $\langle a_i \rangle$. Following with the above example, $[\uparrow_x]$ and $[\downarrow_x]$ are I-case-properties of the spin in the *x*-direction $[S_x]$ in this particular system, where $[S_x]$ is an I-type-property of the U-type-property spin in the *x*-direction $\langle S_x \rangle$, and $[\uparrow_x]$ and $[\downarrow_x]$ are instances of the U-case-properties $\langle \uparrow_x \rangle$ and $\langle \downarrow_x \rangle$, respectively. Again, if the physical, the ontological, and the mathematical languages are distinguished, it should be said that (a) the I-case-properties $[a_i]$ are physically represented by the possible values a_i of the observable A , and (b) the possible values a_i are mathematically represented by the eigenvalues of the self-adjoint operator $\hat{A} = \sum_i a_i \Pi_i$, where Π_i is the eigenprojector corresponding to a_i, such that $\Pi_i = |a_i\rangle\langle a_i|$ in the non-degenerate case.

III) *Bundles*

In the MHI ontology, the ontological correlates of quantum systems are not particles or individuals, but *bundles of properties*. More precisely, a bundle $\mathfrak{B} = \big[[A],[B],[C],\ldots\big]$ is a collection of I-type-properties $[A],[B],[C],\ldots$ corresponding to the U-type-properties $\langle A \rangle, \langle B \rangle, \langle C \rangle, \ldots$. Once again, if the physical, the ontological, and the mathematical languages are distinguished, it should be said that (a) the bundle $\mathfrak{B}$ is physically represented by a system $\mathcal{S}$, which is identified with a space of observables $\mathcal{O}$, and (b) the physical system $\mathcal{S}$ is mathematically represented by the space of operators $\hat{\mathcal{O}}$, or by the Hilbert space $\mathcal{H}$ if $\hat{\mathcal{O}} = \mathcal{B}(\mathcal{H})$. However, this picture differs from that provided by the traditional bundle theory in relevant aspects.

First, in the traditional versions of the bundle theory, an object is the bundle of certain determinate properties, under the implicit assumption that the object's determinable properties are all determinate. For example, a billiard ball is the confluence of a definite value of position, say here, a definite shape, say round, a definite colour, say white, etc.; it is a bundle of determinates. In the quantum domain, by contrast, the Kochen-Specker theorem (Kochen and Specker 1967) proves the impossibility of ascribing precise values to all the observables of a quantum system simultaneously, while preserving the functional relations between commuting observables. From an ontological viewpoint, the theorem challenges the Principle of Omnimode Determination, according to which, in an individual, all determinables are determinate (Wolff 1980, Kant1902): in a quantum system there are always determinables that are not determinate. Then, the system cannot be identified with a bundle of determinate properties. As a consequence, in the MHI ontology, a quantum system is a bundle of determinables, that is, a bundle of I-type-properties, each one of them with its I-case-properties.

Second, in its traditional versions, the bundle theory is a theory about individuals: it is designed to account for individuals without appealing to a substratum on which properties inhere (see, e.g., O'Leary-Hawthorne 1995, French and Bigaj 2024). To this end, some properties must be selected to play the role of the principle that endows individuals with synchronic, diachronic, and transcontextual identification. The MHI view, by contrast, completely dispenses with the ontological category of individual: bundles of properties do not behave as individuals in any way. This means that, when two bundle-systems are combined, the system resulting from the combination is also a bundle. And since bundles are not individuals, there is no principle that preserves their individuality: in the resulting system, the individuality of the original systems is not retained, precisely because they are not individuals at all. In this sense quantum systems are conceived as *non-individual bundles of properties*.

In this ontology, indistinguishability is not a relation between particles because there are no particles at all: indistinguishability is primarily a relation between properties, which, not being individuals, are not subject to any restriction that prevents them from being only numerically different.[9] More precisely: (i) any two I-case-properties $[a_1]$ and $[a_2]$ are indistinguishable, $[a_1] \triangleq [a_2]$, when they are I-case-properties of the same U-case-property $\langle a \rangle$, (ii) any two I-

[9] From a strong empiricist viewpoint, indiscernible properties have been interpreted as the result of measuring the same observable more than once, since the laboratory conditions necessarily change, even if imperceptibly, in the different measurements (see de Barros et al. 2017).

type-properties $\left[A^1\right]$ and $\left[A^2\right]$ are indistinguishable, $\left[A^1\right] \triangleq \left[A^2\right]$, when they are I-type-properties of the same U-type-property $\langle A \rangle$ and their respective I-case-properties are indistinguishable, and (iii) two bundles $\mathfrak{B}^1$ and $\mathfrak{B}^1$ are indistinguishable when their respective I-type-properties are indistinguishable. In short, indistinguishability is primarily a relation between I-case-properties, and derivatively it is a relation between I-type-properties; and from this meaning applied to properties, indistinguishability acquires a precise meaning when applied to bundles.

This general structure of the MHI dissolves rather than solves the problem of quantum indistinguishability. In fact, since quantum systems are not particles, the question can no longer be posed in terms of why a permutation of individual particles does not count as a different distribution. Now, indistinguishability is an internal symmetry of a bundle that was formed by combining indistinguishable bundles (for details, see Lombardi 2025). Nevertheless, here we will not delve into this aspect of the MHI ontology, but we will focus on the logical problem: which logical language can be used to speak about this ontology, given that the vast majority of systems of logic presuppose domains of individuals? A way out of this problem would be to develop a "logic of predicates" in the spirit of the Calculus of Relations proposed by Tarski (1941), in which individual constants and variables are absent. However, this strategy is not entirely legitimate: as we have mentioned, Tarski starts with a traditional set theory (say ZF), and then chooses some relations as primitives and establishes some of their properties as axioms; in this way, individuals apparently "disappear", although they are still implicitly presupposed by the very strategy of the construction of the system.

As explained in Subsection 2.2, Krause's quasi-set theory $\mathfrak{Q}$ was devised by its author with an intended interpretation: it includes two kinds of *Urelements*, M-atoms—classical elements— and m-atoms—indiscernible elements. In this system, m-atoms are conceived as the correlates of elemental particles, which ontologically are non-individuals (see Arenhart et al. 2019), so that the formal relation of indiscernibility represents the physical relation of indistinguishability between elemental particles of the same kind. Nevertheless, precisely due to its formal nature, quasi-set theory can be detached from that intended interpretation and can be applied to items belonging to any ontological category, whenever the logical "behaviour" of those items agrees with the structure of the theory. In particular, quasi-set theory can be applied to a universe of properties, in particular, of I-type-properties, which under certain precise circumstances can be considered as legitimately indistinguishable due to their own ontological nature. In turn, the indistinguishability between bundles is inherited from the indistinguishability of the bundle's components. It can be shown that collections of indistinguishable I-type-properties can be formally characterized as qsets, and can be formed by applying the Axiom of Selection of Indiscernibles to those qsets: a bundle is a qset built with only one element of each one of them. Moreover, qsets of bundle-qsets, with definite quasi-cardinality, can also be defined in order to treat collections of indistinguishable bundles with a definite number of members, in spite of the fact that those members can neither be counted nor labeled (for formal details, see Holik et al. 2022).

Nevertheless, quasi-set theory $\mathfrak{Q}$ is tied to the existence of *Urelemente*: the postulation of M-atoms and m-atoms reveals the persistence of the commitment to items that, although non-individuals, still belong to an ontological category that is not that of property. Therefore, in order to speak of a strict ontology of properties, it is necessary to remove the postulation of atomic elements. In the light of this idea, the quasi-set theory $\mathfrak{Q}^{-}$ was proposed, as explained above, which is not committed to the existence of *Urelemente*. The next subsection will be devoted to show how the theory $\mathfrak{Q}^{-}$ can serve as a metalanguage to speak of the MHI ontology.

4.2.- $\mathfrak{Q}^{-}$ as the metalanguage for the MHI ontology

This subsection will introduce some guidelines to map qsets to the properties postulated by the MHI. The framework presented in this section, based on q-functions, establishes certain restrictions that are necessary for the $\mathfrak{Q}^{-}$ language, conceived as a metalanguage, to be well adapted to the ontology of properties of the MHI.

4.2.1.- Instantiation and determination represented by q-functions

Let us recall that, as explained in the previous subsection, instantiation is the relation between a universal property and its many instances, and determination is the relation between a determinable and its determinates. The two relations are ontological primitives in the MHI ontology. In the $\mathfrak{Q}^{-}$ language, they are represented by *q-functions*. More precisely, four q-functions will be introduced, in which the subindexes '*c*' and '*t*' correspond to case-properties and type-properties respectively.

The first step is to assign (i) a qset $x_{[a]}$ to the I-case-property $[a]$ in such a way that $x_{[a]} \equiv x_{[b]}$ if and only if $[a]$ and $[b]$ are indistinguishable, $[a] \triangleq [b]$, and (ii) a qset $x_{\langle a \rangle}$ to the U-case-property $\langle a \rangle$. The second step is to represent the ontological relation of indistinguishability, ‘$\triangleq$’, by means of the logical relation of indiscerniblity, ‘$\equiv$’, of the language of $\mho^-$.

q-function I_c: ‘being an instance of’ between case properties

I_c : $\left[x_{[a]}\text{: I-case-property } [a]\right] \longrightarrow \left[x_{\langle a\rangle}\text{: U-case-property } \langle a\rangle\right]$

The q-function I_c takes a qset corresponding to an I-case-property as input and returns a qset corresponding to a U-case-property. For example, $I_c\left(x_{[red_1]}\right) = x_{\langle red_1\rangle}$ can be read ‘the I-case-property $[red_1]$ is an instance of the U-case-property $\langle red_1\rangle$ in the ontological language.

Let us recall that, in the MHI ontology, indistinguishability is a primitive relation between I-case-properties: $[a]$ and $[b]$ are indistinguishable, $[a]\triangleq[b]$, if they are instances of the same U-case-property or, at least, of indistinguishable U-case-properties. This ontological fact is represented by the logical fact that I_c is a q-function and, as a consequence, maps indiscernibles onto indiscernibles:

$$x_{[a]} \equiv x_{[b]} \Rightarrow I_c\left(x_{[a]}\right) \equiv I_c\left(x_{[b]}\right)$$

On the other hand, if two U-case-properties are indistinguishable, then all their instances must retain this property. This ontological fact is represented by the following proposition of $\mho^-$:

$$I_c\left(x_{[a]}\right) \equiv I_c\left(x_{[b]}\right) \Rightarrow x_{[a]} \equiv x_{[b]}$$

Therefore, the indiscernibility of the qsets representing I-case-properties is directly linked to the indiscernibility of the qsets representing U-case-properties:

$$x_{[a]} \equiv x_{[b]} \Leftrightarrow I_c\left(x_{[a]}\right) \equiv I_c\left(x_{[b]}\right)$$

From a logical viewpoint, nothing prevents speaking of indiscernible qsets corresponding to indistinguishable U-case-properties. However, from an ontological viewpoint, the idea of indistinguishable universals does not sound natural: there exists a single universal, say, red, with its multiplicity of instances. Therefore, for metaphysical reasons, the qsets assigned to U-case-properties are sets, so that indistinguishability collapses to identity. Therefore, the indiscernibility of the qsets representing I-case-properties leads to the identity of the qsets representing U-case-properties:

$$x_{[a]} \equiv x_{[b]} \Rightarrow I_c\left(x_{[a]}\right) = I_c\left(x_{[b]}\right)$$

q-function D_U : 'being a determination of' between universals

D_U : $\left[x_{\langle a\rangle}\text{: U-case-property } \langle a\rangle\right] \longrightarrow \left[x_{\langle A\rangle}\text{: U-type-property } \langle A\rangle\right]$

The q-function D_U takes a qset corresponding to a U-case-property as input and returns a qset corresponding to a U-type-property. For example, '$D_U\left(x_{\langle 5\,Joules\rangle}\right)=x_{\langle Energy\rangle}$' can be read 'the U-case-property $\langle 5\,Joules\rangle$ is a determination of the the U-type-property $\langle Energy\rangle$ in the ontological language. Moreover, the distinguishable U-case-properties $\langle 5\,Joules\rangle$ and $\langle 6\,Joules\rangle$ are both determinations of the U-type-property $\langle Energy\rangle$.

Since D_U is a q-function, it maps indiscernibles onto indiscernibles:

$$x_{\langle a\rangle} \equiv x_{\langle b\rangle} \Rightarrow D_U\left(x_{\langle a\rangle}\right) \equiv D_U\left(x_{\langle b\rangle}\right)$$

q-function I_t : 'being an instance of' between type properties

I_t : $\left[x_{[A]}\text{: I-type-property } [A]\right] \longrightarrow \left[x_{\langle A\rangle}\text{: U-type-property } \langle A\rangle\right]$

The q-function I_t takes a qset corresponding to an I-tyè-property as input and returns a qset corresponding to a U-type-property. For example, $I_t\left(x_{\left[Energy^1\right]}\right)=x_{\langle Energy\rangle}$ can be read 'the I-type-property $\left[Energy^1\right]$ is an instance of the U-type-property $\langle Energy\rangle$ in the ontological language.

q-function D_I : 'being a determination of' between instances

D_I : $\left[x_{[a]}\text{: I-case-property } [a]\right] \longrightarrow \left[x_{[A]}\text{: I-type-property } [A]\right]$

The q-function D_I takes a qset corresponding to a I-case-property as input and returns a qset corresponding to a I-type-property. For example, '$D_I\left(x_{\left[5\,Joules^1\right]}\right)=x_{\left[Energy^1\right]}$' can be read 'the I-case-property $\left[5\,Joules^1\right]$ is a determination of the the I-type-property $\left[Energy^1\right]$' in the ontological language.

In order to relate the indistinguishability between U-type-properties with the indistinguishability between U-case-properties, the following condition is introduced:

$$x_{\langle A\rangle} \equiv x_{\langle B\rangle} \Leftrightarrow \forall x_{\langle a\rangle} \exists x_{\langle b\rangle}\left(x_{\langle A\rangle} = D_U\left(x_{\langle a\rangle}\right) \wedge x_{\langle B\rangle} = D_U\left(x_{\langle a\rangle}\right) \wedge x_{\langle a\rangle} \equiv x_{\langle b\rangle}\right)$$

This condition expresses, in logical language, the ontological fact that two U-type-properties are indistinguishable if and only if, for every U-case-property that is a determination of one of them, there exists another U-case-property, indistinguishable from the first, that is a determination of the other. As remarked above, this condition admits the existence of indistinguishable universals; nevertheless, if this possibility is to be rejected for metaphysical reasons, the indiscernibility between the qsets corresponding to indistinguishable universals collapses to identity.[10]

Since the indistinguishability between U-case-properties and the indistinguishability between I-case-properties can be expressed by means of the q-function I_c, and the above condition links the indistinguishability between U-type-properties and the indistinguishability between U-case-properties, then the relation between the indistinguishability between U-type-properties and the indistinguishability between the associated I-case properties can also be expressed in the language of $\mathfrak{Q}^-$.

4.2.2.- A strategy to associate logical objects to properties

Up to this point, the assignment of qsets to properties was assumed. However, some account of this assignment is necessary in order to complete the application of the theory $\mathfrak{Q}^-$ to the MHI ontology. Strictly speaking, it is only necessary to explain the assignment of logical objects of the theory to I-case-properties, because the assignment of qsets to the other kinds of properties is determined by the q-functions just introduced. Of course, there are infinite ways to perform this task: here a possible strategy will be proposed.

Let us consider a quantum system $\mathcal{S}$ with a discrete number of observables, each one of which has discrete eigenvalues: in ontological terms, each one of the I-type-properties of the bundle-system has discrete I-case-properties.

- The first step is to assign qsets to the I-type-properties of the bundle-system. Since they are distinguishable, they form a set. If it is assumed that such a set is well ordered, the assignment can proceed as follows. The empty qset $\varnothing$ is assigned to the first element of the set, the qset $\mathcal{P}(\varnothing)$ is assigned to the second element, the qset $\mathcal{P}(\mathcal{P}(\varnothing))$ is assigned to the third element, and so on, such that the qset $\mathcal{P}^{(n-1)}(\varnothing)$ is assigned to the *n*-th element.
- The second step is to distinguish between the I-case-properties of each I-type-property. Given that the above procedure is analogous for any I-type-property, let us consider a generic I-type-property corresponding to $\mathcal{P}^i(\varnothing)$. Since the I-case-properties of this I-type-property are distinguishable, they form a set. If it is assumed that this set is well ordered, the distinction can proceed as follows. A weak singleton $[\varnothing]_{w_1}$ (any of the indiscernibles

[10] It can also be considered that, although universals have identity, they may have many indiscernible qsets associated with them in the formalism, just as a standard real number has a "halo" of indiscernible entities associated with it in the framework of non-standard analysis (Robinson 1974).

compatible with the theory), such that $K\left([\varnothing]_{w_1}, \overline{1}\right)$, is associated to the first element of the set. The qset $[\varnothing]_{w_2}$, such that $K\left([\varnothing]_{w_2}, \overline{1}\right)$, is associated to the second element of the set. The qset w_2 from which the indiscernible elements are extracted can be indiscernible from w_1 ($w_1 \equiv w_2$) as long as w_2 has at least two indistinguishable empty sets, in order to guarantee that the q-cardinality of the weak singleton $[\varnothing]_{w_2}$ is $\overline{2}$. This procedure can be repeated, so that the qset $[\varnothing]_{w_n}$ is associated to the *n*-th element of each eigenvalue set, such that $K\left([\varnothing]_{w_n}, n\right)$. On this basis, the ordered pair $\left[\mathcal{P}^i(\varnothing), [\varnothing]_{w_1}\right]$ is assigned to the first element of the set of I-case properties of the considered I-type property, the ordered pair $\left[\mathcal{P}^i(\varnothing), [\varnothing]_{w_2}\right]$ is assigned to the second element of the set, and so on.

It is worth emphasizing again that the strategy just presented is one among many possible ways of assigning logical objects to each of the I-case-properties of the I-type-properties of the bundle-system. The only requirement is that each procedure selects a unique logical object to be assigned to each of those I-case-properties.

5.- Concluding remarks

The statistical behaviour of quantum systems manifests puzzling features, leading to the so-called problem of indistinguishability, which has been extensively discussed in the literature on the interpretation of quantum mechanics. The quasi-set theory $\mathfrak{Q}$ was originally proposed to provide a proper metalanguage to deal with quantum indistinguishability by including a particular kind of *Urelemente*, the m-atoms, that may be indiscernible without being identical. In turn, the Modal Hamiltonian Interpretation attempts to provide an ontological picture that makes sense of the otherwise puzzling quantum features: the quantum realm turns out to be an ontology deprived from the category of individual object, in which systems are bundles of properties.

Although quasi-set theory $\mathfrak{Q}$, being a formal theory, can be applied to the MHI ontology on the basis of an appropriate interpretation, the inclusion of m-atoms is not conceptually compatible with the idea of an ontology of properties. For this reason, the quasi-set theory $\mathfrak{Q}^-$ has been formulated, which dispenses with any kind of *Urelemente*. The purpose of the present paper has been to introduce the theory $\mathfrak{Q}^-$, not with all its peculiarities, but with a level of detail that is sufficient for its application to the MHI ontology. On this basis, it has been explained how theory $\mathfrak{Q}^-$ can be used as a metalanguage to speak meaningfully of an objectless quantum ontology, where systems do not behave as individuals since they are mere bundles of properties without individuality. Of course, this paper does not intend to be exhaustive from either the logical or the ontological point of view. Nevertheless, we hope that it provides a sufficiently clear starting point to motivate further developments along these lines.[11]

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[11] Both theories $\mathfrak{Q}$ and $\mathfrak{Q}^{-}$ (especially the latter) can be incorporated into the metalanguage of the non-deterministic semantics of Nmatrices (Avron and Zamansky 2010). This makes it possible, on the one hand, to have semantic indeterminism at the level of propositional variables. On the other hand, Nmatrices provide a suitable framework for the indeterministic semantics of the Hilbert lattice of projectors (Jorge and Holik 2020). Therefore, an Nmatrices semantics for the quantum lattice whose metatheory is quasi-set theory can be used as a semantics for a propositional language with indiscernible propositional variables, represented by indiscernible projectors. The above proves that having a language made up of indiscernible projectors is a logical possibility that admits an Nmatrices semantics. If such projectors are to be obtained effectively, $\mathfrak{Q}^{-}$-theory must be used instead of ZF at the level of the object language, in order to construct the Hilbert space. Although this is a work in progress, the first steps in this direction were taken in (Holik et al. 2020).

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